UK Junior Mathematical Olympiad 2003 Solutions

A1 **3** $\sqrt{4 + \sqrt{16 + \sqrt{81}}} = \sqrt{4 + \sqrt{16 + 9}} = \sqrt{4 + 5} = 3.$

A2 **5** There is a 2 by 2 set of cubes in the base which is unpainted and also a single cube in the layer above.

A3 **25** The first ten numbers which are not factors of 720 are: 7, 11, 13, 14, 17, 19, 21, 22, 23 and 25.

A4 **14** Clearly, O is even since it is the units digit of $4 \times N$. Also, O cannot be greater than 2 since that would make $4 \times ON$ a three-digit number.. So O is 2 and N is 3 or 8 since $4 \times N$ has units digit 2. However, 4×28 is not a two-digit number and therefore $N = 3$. The sum is $23 + 23 + 23 + 23 = 92$ and so $G + N + O = 9 + 3 + 2 = 14$.

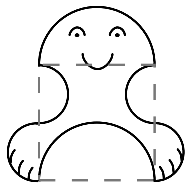
A5 **75m** The fraction of the length of the bridge which is actually over the river is $1 - \left(\frac{1}{3} + \frac{1}{5}\right) = \frac{7}{15}$. Thus $\frac{1}{15}$ of the bridge is 5 metres long giving the bridge length as 75 metres.

A6 $\mathbf{2\frac{1}{2}}$ If the starting number is represented by x, the final number is given by $(2x + 1)/(x - 1)$. So it is necessary to solve $(2x + 1)/(x - 1) = 4$. This gives $2x + 1 = 4x - 4$, which leads to $2x = 5$ and so $x = 2\frac{1}{2}$.

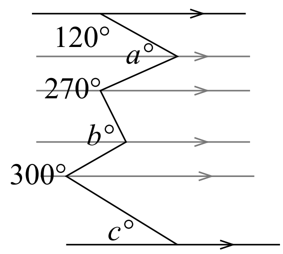
A7 **50** Let the number in the centre be x. Then the 'magic' total is $x + 20$. So the middle number in the top row is $x + 5$ and the middle number in the bottom row is $x - 5$. Thus the sum of the middle column is $(x + 5) + x + (x - 5) = 3x$ but this has to be the same as $x + 20$. So $x = 10$ and the other four empty squares are: 15, 11, 9 and 5 giving the total 50.

Note: the middle number, 10, is one third of the magic total, 30. This is always the case in a 3 × 3 magic square. Can you prove it?

A8 **64 cm²** Placing a square over the creature shows that the bits which stick out of the square can be fitted into the empty bits of the square. The square is 8×8 so the area is 64 cm².



A9 **210** First calculate the two reflex angles shown and draw in four parallel lines. Angles between parallel lines total 180° and the diagram has five pairs of adjacent parallel lines. Thus



$$120 + a + 270 + b + 300 + c = 5 \times 180$$

$$\text{i.e.} \quad a + b + c = 900 - 690 = 210.$$

A10 **80 cm³** Let the smallest dimension of the cuboid, which we will call the height, be h cm. Then the length is $(10 - h)$ cm and the breadth is $(7 - h)$ cm. But the perimeter of the base is 26 cm, so

$$2(10 - h) + 2(7 - h) = 26$$

$$\text{i.e.} \quad 17 - 2h = 13$$

$$\text{i.e.} \quad h = 2.$$

Thus the length and breadth are 8 cm and 5 cm, giving the volume as 80 cm³.

B1 Note that since $p + q^2 + r^3$ is even, p, q and r cannot all be odd nor can just one of them be odd. As 2 is the only even prime number, $p + q^2 + r^3 = 14$ when all three are even. So two of p, q and r must be odd and the third must be 2.

To reduce the possibilities it is best to start with the cube term. Since $7^3 > 200$, r is 2 or 3 or 5.

When $r = 5$, $p + q^2 = 75$. If $p = 2$ then $q^2 = 73$ which is not a square number. But if $q = 2$ then $p = 71$ which is prime.

When $r = 3$, $p + q^2 = 173$. If $p = 2$ then $q^2 = 171$ which is not square. If $q = 2$ then $p = 169 = 13 \times 13$ and so is not prime.

When $r = 2$, $p + q^2 = 192$. So q is an odd prime less than 15. Working down from $q = 13$, $p = 192 - 169 = 23$ which is prime. When $q = 11$, $p = 192 - 121 = 71$ which is prime. When $q = 7$, $p = 192 - 49 = 143 = 11 \times 13$. When $q = 5$, $p = 192 - 25 = 167$ which is prime. Finally when $q = 3$, $p = 192 - 9 = 183$ which is divisible by 3.

Thus there are four possible triples for (p, q, r):

$$(71, 2, 5); (23, 13, 2); (71, 11, 2) \text{ and } (167, 5, 2).$$

B2 Will does not sit by a girl so the three boys are together (with Will in the middle). Vince is between a girl (Yvonne) and the child from Durham (so that is Will). Since Will is from Durham, he is not from Aberdeen so this is the child on the other side of Zac from Will, i.e. Xenia. In turn this means that Yvonne is from Edinburgh and, as Zac is not from Cardiff, he is from Belfast. Finally, therefore, Vince is from Cardiff The names and places are:

Vince	Will	Xenia	Yvonne	Zac
Cardiff	Durham	Aberdeen	Edinburgh	Belfast

B3 (a) The sum of the exterior angles of a polygon is 360°. Each exterior angle of a regular decagon is $360° \div 10 = 36°$. Hence $\angle ABC = 180° - 36° = 144°$.

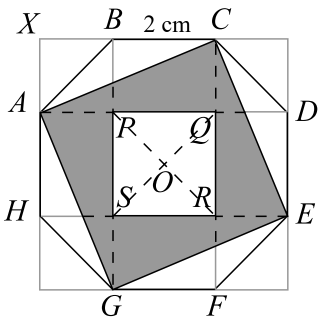
The angles around the point B total 360° so $\angle PBQ = 360° - (144° + 2 \times 90°) = 36°$. But ΔPQB is isosceles, so $\angle BQP = \frac{1}{2}(180° - 36°) = 72°$.

(b) Each exterior angle of the regular polygon is $360°/N$. So $\angle ABC = 180° - 360°/N$. $\angle PBQ = 360°/N$ and $\angle BQP = \frac{1}{2}(180° - 360°/N) = \frac{1}{2}\angle ABC$. Therefore $\angle ABC = 2\angle BQP$ so that the ratio is 2.

Or: Extend PB to X. Then $\angle QBX = 2\angle BQP$, since it is an exterior angle of isosceles triangle PBQ. But $\angle QBX = \angle XBC + 90° = \angle ABC$, ao $\angle ABC = 2\angle BQP$.

B4 (a) Each place in the three-digit integer can be filled in two ways so there are $2 \times 2 \times 2 = 8$ such numbers. They are 222, 223, 232, 233, 322, 323, 332 and 333. Observe that four numbers have a 2 as the hundreds digit and four have a 3, and the same applies to the tens and units digits. So the sum of the numbers equals $4 \times 222 + 4 \times 333 = 4 \times 555 = 2220$.

(b) The same strategy applies here. In this case there are $2^6 = 64$ numbers and their sum is $32(222222 + 333333) = 32 \times 555555 = 2^5 \times 3 \times 5 \times 7 \times 11 \times 13 \times 37$.

B5 

Since $AB = 2$, $AP^2 + BP^2 = 4$ so $AP = BP = \sqrt{2}$. The square with diagonal AB has area 2 and the rectangle $PQCB$ has area $2\sqrt{2}$. So adding these together and dividing by 2 gives the area of ΔAQC as $\frac{1}{2}(2 + 2\sqrt{2})$. So the total shaded area is $2\left(2 + 2\sqrt{2}\right)$.

The octagon can be split into $PQRS$ plus four copies of APB plus four of $PQBC$. Hence the area of the octagon is given by $4 + 4 \times 1 + 4 \times 2\sqrt{2} = 8 + 8\sqrt{2}$. Thus the shaded fraction is

$$\frac{4 + 4\sqrt{2}}{8 + 8\sqrt{2}} = \frac{1}{2}.$$

Or: Note that if square $PQRS$ is divided into four congruent triangles by drawing diagonals PR and QS, which meet at O, then each of these triangles is congruent to triangle ABX. Now triangle AQC is equal in area to triangle AXC i.e. to $\Delta AXB + \Delta ABC$ i.e. to $\Delta POQ + \Delta ABC$. Thus one quarter of the shaded portion of $ABCDEFGH$ is equal in area to one quarter of the unshaded portion, which confirms that the required fraction is indeed a half.

B6 (a) Since the sum of the three numbers 1, 2, 3 is divisible by 3, all numbers made from these digits are divisible by 3. For ab to be divisible by 2, b must be even and hence $b = 2$. So a and c are 1 and 3 or 3 and 1.

There are two numbers of the required form: 123 and 321.

(b) As above, b must be even. Also d must be even for $abcd$ to be divisible by 4, so the first three digits are either 1, 2, 3 or 1, 4, 3 in some order. However, only 1, 2, 3 allow abc to be divisible by 3. Thus $b = 2$ and $d = 4$. For $abcd$ to be divisible by 4 its final pair of digits must be divisible by 4. But if $c = 1$, $cd = 14$ and if $c = 3$, $cd = 34$, neither of which has a factor of 4. Thus there are no numbers of the required form.

(c) Since $abcde$ is to have 5 as a factor, it must end in 5. This means that $abcd$ has to satisfy the conditions of (b) which means that in this case also there are no numbers of the required form.

(d) As above, $e = 5$ and b, d and f are all even, so a and c are 1 and 3 (or 3 and 1). Since none of 143, 163, 341 and 361 is divisible by 3, $b = 2$. So any such numbers are of the form $a2cd5f$. But from (b) $d \neq 4$, so the form becomes $a2c654$. Numbers of this form are even and divisible by 3 (since $1 + 2 + 3 + 4 + 5 + 6 = 21$) and hence are divisible by 6. So the only remaining requirement is that $a2c6$ is divisible by 4. But this is true when $c = 1$ and also when $c = 3$. So there are two numbers of the required form: 123654 and 321654.